Estimating the Value-at-Risk from High-frequency Data

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Abstract:
We present two alternative approaches for estimating VaR. Both approaches are based on the observation that each trading day is very diverse and we can observe K different phases of the trading day. We can not observe from which of the K phases our observations r_t are. Therefore, we apply Gibbs sampler to estimate parameters from our data. In the latter approach, we apply Dubins and Schwarz theorem (Kallenberg, 2000), which allows us to re-scale our portfolio returns r_t and to get normal distributed returns $r_{J_\tau} \sim N(0, J_\tau)$. To verify our approaches, we make an empirical application.

Key words: Data augmentation; Quadratic variation; Time changed Brownian motion and Gibbs sampler.

JEL Classifications: C15, C53.

1 Introduction

There are many financial institutions that have embraced Value-at-Risk (VaR) as the instrument to assess information about their portfolio positions. To calculate the VaR, banks can choose between a historical simulation, a variance method and a Monte Carlo simulation. The first technique does not require any distributional assumptions and to compute VaR it uses essentially only the empirical distribution of the portfolio returns. The second approach assumes that our portfolio returns r_t follow a particular distribution $f(r_t)$ with the cumulative distribution function (cdf) $F(r_t)$. In the ideal case, the cdf $F(r_t)$ has an analytical solution for the inverse cumulative distributed function $F^{-1}(u)$. In the latter approach, both previous approaches are combined together. Many financial institutions assume that their portfolio returns r_t follow one of the Lévy stable distributions. However, there are many external factors such as macroeconomic shocks, trading behavior - herding or executing large portfolios which influence the portfolio returns r_t. Therefore, the assumption that the returns r_t are generated by one distribution is not correct. Moreover, these factors cause that the VaR is overestimated, which has a negative impact on the bank's profit. We introduce two alternative approaches to calculate the Value-at-Risk for the portfolio that cope with these external shocks. The first approach is based on the fact that the trading day is divided into K different phases. We can not observe

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from which K phases our observations r_t are. Therefore, we apply Gibbs sampler to estimate parameters from our data. In the second approach, we apply Dubins and Schwarz theorem (Kallenberg, 2000), which allows us to re-scale our portfolio returns r_t and to get the normal distributed returns $r_{J_\tau} \sim N(0, J_\tau)$.

The paper is organized as follows. In Section 2.1, we present well-known variance method Student-*t* Distribution, which we use it as our benchmark for our proposal techniques. In the Section 2.2, we present the Gibbs sampler algorithm for sampling from the mixture of Gaussian distributions. Since one of the main properties of the high-frequency returns is the heavy-tailed distribution, Section 2.3 is concerned with the formulation of the time change techniques represent by the Dubins and Schwarz theorem (Kallenberg, 2000), and we will use it in the mixture of Gaussian model. After formulating the methods for computing Value-at-Risk in Section'2.3, we apply these techniques to the market data and compare the results in Section 3.

2 Methods for Computing Value-at-Risk

In this Section 2, we present three different methods for computing VaR. First approach is a standard variance method for computing VaR. The second approach assume that the portfolio returns r_t follow one of the K differ distributions. The latter technique also allow us to cope with the many properties of high frequency data.

2.1 Student-t Distribution

Many financial institutions use variance technique as the standard approach to estimate VaR of the portfolio. We use this method as a benchmark for two approaches presented in the Sections 2.2 and 2.3. Now, we assume that the portfolio returns r_t follow a t-distribution, when the log likelihood function equals

$$\begin{aligned}\mathscr{L} =& T\left[\log\Gamma\left(\frac{\nu+1}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{1}{2}\log\pi\nu - \log\sigma\right] - \\ & \frac{\nu+1}{2}\sum_{t=1}^{T}\log\left(1+\left(\frac{r_t-\mu}{\sigma\sqrt(\nu)}\right)^2\right)\end{aligned} \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, $\mu \in \mathbb{R}$, $\sigma > 0$ is the variance and $\nu > 0$ is the number of degrees of freedom. The Value-at-Risk can be expressed as follows:

$$VaR = -W_0\left(e^{\mu+\gamma F_\nu^{-1}(p)} - 1\right), \tag{2}$$

where W_0 is the initial value of the portfolio and $F_\nu^{-1}(p)$ is the cumulative distribution function of standardized t-distributed random variable.

2.2 Mixture of Gaussians

More suitable approach, which better captures variability of the portfolio, is based on the idea that the i-th return is from different k distribution. To estimate VaR from these data, we use Gibbs sampler which can sample from the full conditionals distributions. We consider portfolio returns $r_1, \ldots, r_n$ defined by the equation (13) stem for one of K population $N(\mu_k, \frac{1}{\tau_k})$, where $k = 1, \ldots, K$. The mixture distribution for $i-$th observation is:

$$f(y_i) = \sum_{k=1}^{K} \pi_k \phi_{(\mu_k, \frac{1}{\tau_k})}(y_i). \tag{3}$$

where $P(R_i = k) = \pi_k$. In a Bayesian analysis a suitable prior distributions for the unknown parameters μ_k, τ_k and π_k are

$$\begin{aligned} \mu_k \sim & N(\mu_0, \frac{1}{\tau_0}) \\ \tau_k \sim & \Gamma(\alpha, \beta) \\ \pi_1, \ldots, \pi_K \sim & f_{(\alpha_1, \ldots, \alpha_K)}(\pi_1, \ldots, \pi_K) \propto \prod_{k=1}^{K} \pi_k^{\alpha_k - 1}, \quad \pi \geq 0, \quad \sum_{k=1}^{K} \pi_k = 1. \end{aligned} \tag{4}$$

For the simplicity, we will assume that the number of populations K is known. In the case, where the number of components is variable, the reversible jump algorithm can be used. To write out the posterior distributions for the unknown variables μ_k, τ_k and π_k given data $R_1, \ldots, R_n$, we suggest auxiliary variables $Z_1, \ldots, Z_n$:

$$\mathbb{P}(Z_i = k) = \pi_k \qquad \text{and} \qquad R_i | Z_i = k \sim N(\mu_k, 1/\tau_k). \tag{5}$$

Thus the posterior distribution is

$$\begin{aligned} & f(r_1, \ldots, r_n, z_1, \ldots, z_n, \mu_1, \ldots, \mu_K, \tau_1, \ldots, \tau_K, \pi_1, \ldots, \pi_K) \propto \\ & \Big(\prod_{i=1}^{n} \pi_{z_i} \exp\Big(\frac{-\tau_{z_i}(r_i - \mu_{z_i})^2}{2} \Big) \Big) \Big(\prod_{k=1}^{K} \exp\Big(\frac{-\tau_0(\mu_k - \mu_0)^2}{2} \Big) \Big) \\ & \Big(\prod_{k=1}^{K} \tau_k^{\alpha - 1} \exp(-\beta \tau_k) \Big) \Big(\prod_{k}^{K} \pi_k^{\alpha_k - 1} \Big). \end{aligned} \tag{6}$$

Now, we can simply derive full conditional distributions given $R_1, \ldots, R_n$

$$\mu_k | \boldsymbol{R}, \boldsymbol{\tau}, \boldsymbol{\pi}, \boldsymbol{Z} \quad \sim \quad N\left(\frac{\sum_{i=1}^{n} R_i \tau_i \mathbb{I}_{[Z_i = 1]} + \tau_0 \mu_0}{\sum_{i=1}^{n} \tau_i \mathbb{I}_{[Z_i = 1]} + \tau_0}, \frac{1}{\sum_{i=1}^{n} \tau_i \mathbb{I}_{[Z_i = 1]} + \tau_0} \right) \tag{7}$$

$$\tau_k | \boldsymbol{R}, \boldsymbol{\mu}, \boldsymbol{\pi}, \boldsymbol{Z} \quad \sim \quad Gamma\Big(\alpha + \frac{\sum_{i=1}^{n} \mathbb{I}_{[Z_i = 1]}}{2}, \beta + \frac{1}{2} \sum_{i=1}^{n} (\mathbb{I}_{[Z_i = 1]} R_i - \mu_i \mathbb{I}_{[Z_i = 1]}) \Big) \tag{8}$$

$$\pi_k | \boldsymbol{R}, \boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{Z} \quad \sim \quad \text{Dirichlet}\Big(\alpha_1 + \sum_{i=1}^{n} \mathbb{I}_{[Z_i = 1]}, \ldots, \alpha_K + \sum_{i=1}^{n} \mathbb{I}_{[Z_i = K]} \Big) \tag{9}$$

$$\mathbb{P}(Z_i = k|\boldsymbol{R}, \boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{\pi}) = \frac{\pi_k \phi_{(\mu_k, \frac{1}{\tau_k})}(y_i)}{\sum_{j=1}^{K} \phi_{(\mu_j, \frac{1}{\tau_j})}(y_i)} \tag{10}$$

where the following are vectors $\boldsymbol{R}_{n\times1}$, $\boldsymbol{\mu}_{K\times1}$, $\boldsymbol{\tau}_{K\times1}$, $\boldsymbol{\pi}_{K\times1}$, and $\boldsymbol{Z}_{n\times1}$. To get a a sample from the posterior distribution of $u_1,\dots,u_K$, $\tau_1,\dots,\tau_K$ and $\pi_1,\dots,\pi_K$ given observations $R_1,\dots,R_n$ using Gibbs sampler algorithms:

Gibbs sampler

Starting with $\mu_1^{(0)},\dots,\mu_K^{(0)},\tau_1^{(0)},\dots,\tau_K^{(0)},\pi_1^{(0)},\dots,\pi_K^{(0)}$ iterate the following steps for $t = 1,2,\dots,T$

1. For $i = 1,\dots,n$:

   Draw $Z_i^{(t)}$ from the distribution (10).

2. For $k = 1,\dots,K$:

   Draw $\mu_k^{(t)}$ and $\tau_k^{(t)}$from the distributions (7) and (8) respectively.

3. Draw $(\pi_1^{(t)},\dots,\pi_K^{(t)})$ from the distribution (9).

2.3 Mixture of Gaussian with the Time Change

The approach presented in the Section (2.2) is able to 'divide' trading day into to K different phases and therefore the VaR is less overestimated. Nevertheless, approximation of returns r_t by normal distributions does not take into account many properties of high frequency data. To overcome the shortage of the normal distribution, we introduce time change techniques, which allow us to scale data r_t and to get returns of portfolio r_τ follow the normal distribution. Substitution for this obtain property of r_τ, we will assume that the price process $(M_t)_{t\geq0}$ is continuous local $\mathscr{F}$-martingale. In this Section (2.3), we will work on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with filtration which complies with $\mathscr{F}_0 \supseteq \{N \in \mathscr{F}_\infty; \mathbb{P}(N) = 0\}$ and $\mathscr{F}_{t+} \equiv \bigcap_{v>t} \mathscr{F}_v$ for all $t \geq 0$. $W(\cdot)$ is an 1-dimensional Brownian motion.

Definition 2.3.1 (Time change)**.** *A family of random variables $\tau = (\tau(s))_{s\geq0}$ is said to be a random change of time, if*

1. *$\tau = (\tau(s))_{s\geq0}$ is a nondecreasing, right-continuous family of $[0,\infty]$-valued random variables $\tau(s)$, $s \geq 0$;*

2. *for all $s \geq 0$ the random variables $\tau(s)$ are stopping times with respect to the filtration $(\mathscr{F}_t)$.*

Definition 2.3.2 (Subordinator)**.** *The change of time $\tau = (\tau(s))_{s\geq0}$ bears the name of subordinator, if this random process τ on the interval $[0,\infty]$ is a Lévy process.*

Definition 2.3.3 (Quadratic variation). *Let M be an continuous local $\mathscr{F}$-martingale in $\mathbb{R}$, and fix any $t > 0$ and a sequence of partitions $0 = t_{n,0} < t_{n,1} < \ldots < t_{n,k_n} = t$, $n \in \mathbb{N}$, such that $h_n \equiv \max_k(t_{n,k} - t_{n,k-1}) \to 0$. Then*

$$[M]_n = \sum_k (M_{t_{n,k}} - M_{t_{n,k-1}})^2 \tag{11}$$

is a quadratic variation of the continuous local $\mathscr{F}$-martingale.
Next, we present theorem, which has an enormous application in the financial mathematics.
Theorem 2.3.1 (Dambis, Dubins and Schwarz). *Let M be an continuous local $\mathscr{F}$-martingale in $\mathbb{R}$ with $M_0 = 0$ and define*

$$\tau_s = \inf\{t \geq 0; [M]_{t>s}\}, \quad \mathscr{G}_s = \mathscr{F}_{\tau_s}, \quad s \geq 0. \tag{12}$$

Then there exists in $\mathbb{R}$ a Brownian motion B with respect to a standard extension of $\mathscr{G}$, such that a.s. $B = M \circ \tau$ on $[0, [M]_\infty]$ and $M = B \circ [M]$.

Proof. See (Kallenberg, 2000). □

To apply theorem 2.3.1 on our data r_t, we present suitable estimator for quadratic variation $[M]_t$. We assume fixed interval T of length $h > 0$, then the portfolio returns of the t such interval are defined as

$$r_t = M_{th} - M_{(t-1)h}, \quad t = 1, 2, \ldots, T. \tag{13}$$

During the interval h, we can also compute J intra h-returns. These are defined, for the t period as

$$r_{j,t} = M_{\left((t-1)h + \frac{hj}{J}\right)} - M_{\left((t-1)h + \frac{h(j-1)}{J}\right)}, \quad j = 1, 2, \ldots, J. \tag{14}$$

Then we can define variable, which will be used as the estimator of the quadratic variance $[M]_t$.
Definition 2.3.4. *The realized variance during the period h is defined as*

$$[\hat{M}_J]_t = \sum_{j=1}^{J} r_{j,t}^2. \tag{15}$$

For all M continuous local $\mathscr{F}$-martingale following theorem (2.3.2) holds.
Theorem 2.3.2.

$$[\hat{M}_J]_t \xrightarrow{P} [M]_t \qquad as \quad J \to \infty. \tag{16}$$

Proof. See (Andersen, 2001). □

Let $J_\tau = \inf\{j > 0 : [\hat{M}_J]_t \geq h\}$ be a stopping time. Then portfolio returns r_t stopping in the time J_τ are normally distributed $N(0, J_\tau)$. Now, we can use this time change techniques to scale our data and to apply Gibbs sampler algorithm derived in the Section 2.2 to estimate VaR for our portfolio W_0.

3 Empirical Application to Market Data

In this section we take look at real data and show that both the mixture distributions and mixture distributions with time change are able to better estimate VaR of portfolio. We work with Bund futures prices (i.e. futures on German government bonds with maturity 8.5 to 10.5 years) for May 2010 and we consider intraday returns in ticks over hour time period.

A likelihood ratio test (Kupiec, 1995) is used to reveal whether a VaR model is to be rejected or not. Let N be the number of times the portfolio loss is worse than the true VaR in a sample of size T. Then the number of default VaR follows binomial distribution i.e. $N \sim \mathscr{B}(T,p)$. Then the null hypothesis is $H_0 : \frac{N}{T} = p$ with the statistics

$$LR = 2\left[\log\left(\left(\frac{N}{T}\right)^N\left(1-\frac{N}{T}\right)^{T-N}\right) - \log\left(p^N(1-p)^{T-N}\right)\right]. \tag{17}$$

3.1 Results for Real Data

We assumed that our data can be divided into three different clusters. Therefore, both models $\mathscr{M}_2$ and $\mathscr{M}_3$ have three mixture components $K = 3$ with different estimating parameters. These parameters were estimated on the 70% of all observations. Then we applied a likelihood ratio test (Kupiec, 1995) on the rest of observations with the following results:

Tab. 1: Failure rates of the models $\mathscr{M}_1$-t-distribution, $\mathscr{M}_2$-mixture distribution and $\mathscr{M}_3$-mixture distribution with the time change

Left tail	Failure rate - $\mathscr{M}_1$	Failure rate - $\mathscr{M}_2$	Failure rate - $\mathscr{M}_3$
5.00%	66	61*	49*
1.00%	31	15*	10*
0.10%	3*	8	0*
0.01%	2	5	0*

Source: Author computation.

As the table 1 indicates, Kupiec back-test lead to rejection of the model $\mathscr{M}_1$ for the 5%, 1% and 0.01% levels. For the mixture model $\mathscr{M}_2$, we see that the failure rate for the left tails 5% and 1% is lower and the Kupiec back-test does not reject model for thse tails. Finally, model $\mathscr{M}_3$ is not rejected by the test for any of the left tails.

4 Conclusion

We presented and compared three approaches on our market data used in computing Value-at-Risk of portfolio W_0. Except the first approach, which is just our benchmark for other approaches, we presented data augmentation techniques that helped us to simulate from the K different distributions and to better capture the variability of the financial market. Further, we also used a time change technique (Kallenberg, 2000) which allowed us to rescale our high frequency returns r_t and to get the normally distributed portfolio returns $r_{J_\tau} \sim N(0, J_\tau)$. Finally, our empirical application revealed that the rescaled returns r_{J_t} together with the assumption about the K different phases of the trading day give us the best result for the estimated VaR.

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